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LOCAL WELL-POSEDNESS OF PERIODIC FIFTH ORDER KDV TYPE EQUATIONS

YI HU AND XIAOCHUN LI

ABSTRACT. In this paper, the local well-posedness of periodic fifth order dispersive equation with nonlinear term $P_1(u)\partial_x u + P_2(u)\partial_x u\partial_x u$. Here $P_1(u)$ and $P_2(u)$ are polynomials of u . We also get some new Strichartz estimates.

1. INTRODUCTION

In this paper, we consider the following Cauchy problem on the fifth order dispersive equations:

$$(1.1) \quad \begin{cases} \partial_t u + \partial_x^5 u + P_1(u)\partial_x u + P_2(u)\partial_x u\partial_x u = 0 \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, t \in \mathbb{R}. \end{cases}$$

where P_1 and P_2 are polynomials.

Theorem 1.1. *The Cauchy problem (1.1) is locally well-posed provided that the initial data $\phi \in H^s$ for $s > 1$.*

The index $s = 1$ is sharp for (1.1) to be well-posed (See Section 2). If the nonlinear term $P_2(u)\partial_x u\partial_x u$ in (1.1) is removed, then we may get a better regularity condition on s . More precisely, we have

Theorem 1.2. *Let P_1 be a polynomial of degree $k \geq 2$. Then the Cauchy problem*

$$(1.2) \quad \begin{cases} \partial_t u + \partial_x^5 u + P_1(u)u_x = 0 \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, t \in \mathbb{R}. \end{cases}$$

is locally well-posed if the initial data $\phi \in H^s$ for $s > 1/2$.

Even for $P_1 = 0$ in (1.1), the sharp regularity condition is still $s \geq 1$. In this case, the following well-posedness can be established.

Theorem 1.3. *The Cauchy problem*

$$(1.3) \quad \begin{cases} \partial_t u + \partial_x^5 u + P_2(u)\partial_x u\partial_x u = 0 \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, t \in \mathbb{R}. \end{cases}$$

is locally well-posed provided that the initial data $\phi \in H^s$ for $s > 1$.

Remark 1.1. *If P_2 is a polynomial of degree 0 or 1, then Theorem 1.3 holds for the endpoint $s = 1$.*

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If P_1 is a polynomial of degree 1, the local well-posedness of (1.2) for $s > 0$ was proved by Bourgain in [3]. Moreover, in the same paper, Bourgain proved (1.1) is locally well-posed if s is sufficiently large. Only the lower order derivative of u is allowed in the nonlinear term of (1.1), because the ill-posedness of

$$(1.4) \quad \begin{cases} \partial_t u + \partial_x^5 u + u^2 \partial_x^2 u = 0 \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, t \in \mathbb{R}. \end{cases}$$

even for smooth initial data ϕ was observed by Bourgain in [3]. Theorem 1.1 is still true even if polynomials P_1 and P_2 are replaced by sufficiently smooth functions. One may utilize the ideas in [8] to obtain this result. For technical simplicity, in this paper, we do not provide the details on the general smooth nonlinear terms. In what follows, we only need to prove Theorem 1.2 and Theorem 1.3, since Theorem 1.1 can be done similarly. The higher order dispersive equations associated with smooth nonlinear terms will be studied in our next paper.

As we did in [8], in order to prove Theorem 1.2 and Theorem 1.3, we need to build up some Strichartz inequalities. Let $K_{d,p,N}$ be the best constant satisfying

$$(1.5) \quad \left\| \sum_{n=-N}^N a_n e^{2\pi i t n^d + 2\pi i x n} \right\|_{L_{x,t}^p(\mathbb{T} \times \mathbb{T})} \leq K_{d,p,N} \left(\sum_{n=-N}^N |a_n|^2 \right)^{\frac{1}{2}}.$$

This is the periodic Strichartz inequality associated to higher order dispersive equations. First L^6 estimate can be established.

Theorem 1.4. *Let $K_{d,p,N}$ be defined as in (1.5). If d is odd, then for any $\varepsilon > 0$, there exists a constant C independent of N such that*

$$(1.6) \quad K_{d,6,N} \leq C N^\varepsilon.$$

Second, for large p we have sharp estimates (up to a factor of N^ε).

Theorem 1.5. *Let $K_{d,p,N}$ be defined as in (1.5). If $p \geq p_0$, then for any $\varepsilon > 0$, there exists a constant C independent of N such that*

$$(1.7) \quad K_{d,p,N} \leq C N^{\frac{1}{2} - \frac{(d+1)}{p} + \varepsilon}.$$

Here p_0 is given by

$$(1.8) \quad p_0 = \begin{cases} (d-2)2^d + 6 & \text{if } d \text{ is odd} \\ (d-1)2^d + 4 & \text{if } d \text{ is even} \end{cases}$$

In terms of the language of discrete restriction, Theorem 1.5 is equivalent to

$$(1.9) \quad \sum_{n=-N}^N \left| \widehat{f}(n, n^d) \right|^2 \leq C N^{1 - \frac{2(d+1)}{p} + \varepsilon} \|f\|_{p'}^2, \text{ for } p \geq p_0.$$

Here f is a periodic function on \mathbb{T}^2 , \widehat{f} is Fourier transform of f on \mathbb{T}^2 , $d \geq 3$ is an integer, $p \geq 2$ and $p' = p/(p-1)$. The $d = 2$ case was investigated by Bourgain [1]. It was proved by Bourgain in [2] that $K_{d,4,N} \leq C$ and $K_{3,6,N} \leq N^\varepsilon$. The results for large p and $d = 3$ were established in [8]. The main ideas utilized in this paper come from [7] and [8].

2. TWO COUNTER EXAMPLES

In this section, we give two examples showing that the indices $\frac{1}{2}$ in Theorem 1.2 and 1 in Theorem 1.3 are sharp. More precisely, one has analytic ill-posedness in (1.2) if $s < \frac{1}{2}$ and in (1.3) if $s < 1$. The two examples provided in this section are simple modifications of those in [3] and [4].

First consider (1.2) and take $P_1(u) = u^2$. Define the iterates $u^{(0)}$ and $u^{(1)}$ by

$$(2.10) \quad \partial_t u^{(0)} + \partial_x^5 u^{(0)} = 0, \quad u^{(0)}(x, 0) = \phi(x),$$

$$(2.11) \quad \partial_t u^{(1)} + \partial_x^5 u^{(1)} + (u^{(0)})^2 \partial_x u^{(0)} = 0, \quad u^{(1)}(x, 0) = \phi(x).$$

In order for (1.2) to be locally wellposed, we must have

$$(2.12) \quad \sup_{0 < t < \delta} \|u^{(1)}\|_{H_x^s} < \infty$$

for some positive small δ .

Let N be a (large) positive integer and let

$$(2.13) \quad \phi(x) = \frac{\varepsilon}{N^s} e^{iNx} + \frac{\varepsilon}{N^s} e^{-iNx}$$

be a specific initial value. Obviously $\phi \in H^s$. Then $u^{(0)}$ in (2.10) equals

$$(2.14) \quad \frac{\varepsilon}{N^s} e^{iNx} e^{-iN^5 t} + \frac{\varepsilon}{N^s} e^{-iNx} e^{iN^5 t}.$$

Thus in (2.11), the nonlinear term $(u^{(0)})^2 \partial_x u^{(0)}$ can be expressed as

$$(2.15) \quad i\varepsilon^3 N^{1-3s} \left(e^{iNx} e^{-iN^5 t} - e^{-iNx} e^{iN^5 t} + e^{i3Nx} e^{-i3N^5 t} - e^{-i3Nx} e^{i3N^5 t} \right).$$

A simple calculation, via a use of (2.15) and Duhamel's formula, allows us to represent $u^{(1)}$ as

$$(2.16) \quad u^{(1)}(x, t) = (\varepsilon N^{-s} - i\varepsilon^3 N^{1-3s} t) e^{-iN^5 t} e^{iNx} + \dots.$$

The remaining term " \dots " in (2.16) is of the form

$$(2.17) \quad \sum_{\ell \neq 1} C_\ell e^{i\ell Nx} f_\ell(t)$$

where f_ℓ 's are functions of the time variable t only. Henceforth using the definition of H^s norm, we obtain

$$(2.18) \quad \|u^{(1)}\|_{H_x^s} \geq C\varepsilon^3 N^{1-2s} t.$$

This shows s must be at least $\frac{1}{2}$, since otherwise $\|u^{(1)}\|_{H_x^s}$ would blow up as N goes to infinity, which contradicts (2.12). This example can be simply modified for the case when $P_1(u) = u^{2k}$. Hence $s = 1/2$ is the best regularity condition for (1.2) to be well-posed.

Next we consider (1.3) and take $P_2(u) = u$. Define the iterates $u^{(0)}$ and $u^{(1)}$ by

$$(2.19) \quad \partial_t u^{(0)} + \partial_x^5 u^{(0)} = 0, \quad u^{(0)}(x, 0) = \phi(x),$$

$$(2.20) \quad \partial_t u^{(1)} + \partial_x^5 u^{(1)} + u^{(0)} (\partial_x u^{(0)})^2 = 0, \quad u^{(1)}(x, 0) = \phi(x).$$

Similarly, local well-posedness implies (2.12). Take the same initial value as in (2.13), so for (2.19) we get the same $u^{(0)}$ as in (2.14). Thus in (2.20), the nonlinear term $u^{(0)} (\partial_x u^{(0)})^2$ can be expressed as

$$(2.21) \quad u^{(0)} (\partial_x u^{(0)})^2 = \varepsilon^3 N^{2-3s} \left(e^{iNx} e^{-iN^5 t} + e^{-iNx} e^{iN^5 t} - e^{i3Nx} e^{-i3N^5 t} - e^{-i3Nx} e^{i3N^5 t} \right).$$

Again by (2.21) and Duhamel's formula, one may represent $u^{(1)}$ as

$$(2.22) \quad u^{(1)}(x, t) = (\varepsilon N^{-s} - i\varepsilon^3 N^{2-3s} t) e^{-iN^5 t} e^{iNx} + \dots$$

Here " \dots " is of the form (2.17). From (2.22), we get immediately

$$(2.23) \quad \|u^{(1)}\|_{H_x^s} \geq C\varepsilon^3 N^{2-2s} t,$$

which implies $s \geq 1$, since otherwise $\|u^{(1)}\|_{H_x^s}$ would blow up as N goes to infinity, which contradicts (2.12). This example can be also generalized to the general polynomial case.

3. PROOF OF THEOREM 1.4

Via a direct calculation, we reduce the problem to count the number of integral solutions of

$$(3.1) \quad \begin{cases} n_1 + n_2 + n_3 = A \\ n_1^d + n_2^d + n_3^d = B. \end{cases}$$

Here A, B are fixed constants such that $|A| \leq 3N$ and $|B| \leq 3N^d$. Write $n_3 = A - n_1 - n_2$ in the second equation and we then obtain

$$(3.2) \quad n_1^d + n_2^d + (A - n_1 - n_2)^d = B.$$

Applying the binomial theorem, we get

$$(3.3) \quad -\sum_{k=1}^{\frac{d-1}{2}} C(d, k) n_1^k n_2^k (n_1^{d-2k} + n_2^{d-2k}) + \sum_{k=1}^{d-1} C(d, k) A^{d-k} (-1)^k (n_1 + n_2)^k = B - A^d.$$

Here $C(d, k)$ stands for the binomial coefficient. Since $d - 2k$ is odd, $n_1 + n_2$ is a factor of the left hand side of (3.3). Henceforth we have

$$(3.4) \quad (n_1 + n_2) | (B - A^d).$$

By symmetry, we get immediately that $n_1 + n_2$, $n_2 + n_3$ and $n_1 + n_3$ are divisors of $B - A^d$. Therefore Theorem 1.4 follows since there are at most N^ε divisors of $B - A^d$. This completes the proof of Theorem 1.4.

In the end of this section, let us state a useful theorem on L^4 estimate, proved by Bourgain in [2]. A consequence of Theorem 3.1 is, in terms of $X_{s,b}$ defined as in Definition 7.1, $X_{0, \frac{d+1}{4d}} \subset L_{loc}^4$.

Theorem 3.1. *For any function f on \mathbb{T}^2 ,*

$$(3.5) \quad \|f\|_4 \leq C \left(\sum_{m, n \in \mathbb{Z}} (1 + |n - m^d|)^{\frac{d+1}{2d}} |\widehat{f}(m, n)|^2 \right)^{1/2}.$$

4. PROOF OF THEOREM 1.5

The argument in this section is a modification of those in [7] and [8]. For the sake of self-containedness, we present all details here. To prove Theorem 1.5, we need to introduce a level set. Let F_N be a periodic function on \mathbb{T}^2 given by

$$(4.1) \quad F_N(x, t) = \sum_{n=-N}^N a_n e^{2\pi i n x} e^{2\pi i n^d t},$$

where $\{a_n\}$ is a sequence with $\sum_n |a_n|^2 = 1$ and $(x, t) \in \mathbb{T}^2$. For any $\lambda > 0$, set a level set E_λ to be

$$(4.2) \quad E_\lambda = \{(x, t) \in \mathbb{T}^2 : |F_N(x, t)| > \lambda\}.$$

To obtain the desired estimate for the level set, let us first state a lemma on Weyl's sums.

Lemma 4.1. *Suppose that $t \in \mathbb{T}$ satisfies $|t - a/q| \leq 1/q^2$, where a and q are relatively prime. Then if $q \geq N^{d-1}$,*

$$(4.3) \quad \left| \sum_{n=1}^N e^{2\pi i t n^d + 2\pi i P(n)} \right| \leq C N^{-d2^{1-d}+1+\varepsilon} q^{2^{1-d}}.$$

Here P is a real polynomial of degree no more than $d-1$, and the constant C is independent of t , P , a/q and N .

The proof of Lemma 4.1 relies on Weyl's squaring method. See [9] or [11] for details. Also we need the following lemma proved in [1].

Lemma 4.2. *For any integer $Q \geq 1$ and any integer $n \neq 0$, and any $\varepsilon > 0$,*

$$\sum_{Q \leq q < 2Q} \left| \sum_{a \in \mathcal{P}_q} e^{2\pi i \frac{a}{q} n} \right| \leq C_\varepsilon d(n, Q) Q^{1+\varepsilon}.$$

Here \mathcal{P}_q is given by

$$(4.4) \quad \mathcal{P}_q = \{a \in \mathbb{N} : 1 \leq a \leq q \text{ and } (a, q) = 1\}$$

and $d(n, Q)$ denotes the number of divisors of n less than Q and C_ε is a constant independent of Q, n .

Proposition 4.1. *Let K_N be a kernel defined by*

$$(4.5) \quad K_N(x, t) = \sum_{n=-N}^N e^{2\pi i t n^d + 2\pi i x n}.$$

For any given positive number Q with $N^{d-1} \leq Q \leq N^d$, the kernel K_N can be decomposed into $K_{1,Q} + K_{2,Q}$ such that

$$(4.6) \quad \|K_{1,Q}\|_\infty \leq C_1 N^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}}.$$

and

$$(4.7) \quad \|\widehat{K_{2,Q}}\|_\infty \leq \frac{C_2 N^\varepsilon}{Q}.$$

Here the constants C_1, C_2 are independent of Q and N .

Proof. We can assume that Q is an integer, since otherwise we can take the integer part of Q . For a standard bump function φ supported on $[1/200, 1/100]$, we set

$$(4.8) \quad \Phi(t) = \sum_{Q \leq q \leq 5Q} \sum_{a \in \mathcal{P}_q} \varphi\left(\frac{t - a/q}{1/q^2}\right).$$

Clearly Φ is supported on $[0, 1]$. We can extend Φ to other intervals periodically to obtain a periodic function on \mathbb{T} . For this periodic function generated by Φ , we still use Φ to denote it. Then it is easy to see that

$$(4.9) \quad \widehat{\Phi}(0) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{\mathcal{F}_{\mathbb{R}} \varphi(0)}{q^2} = \sum_{q \sim Q} \frac{\phi(q)}{q^2} \mathcal{F}_{\mathbb{R}} \varphi(0)$$

is a constant independent of Q . Here ϕ is Euler phi function, and $\mathcal{F}_{\mathbb{R}}$ denotes Fourier transform of a function on \mathbb{R} . Also we have

$$(4.10) \quad \widehat{\Phi}(k) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{1}{q^2} e^{-2\pi i \frac{a}{q} k} \mathcal{F}_{\mathbb{R}} \varphi(k/q^2).$$

Applying Lemma 4.2 and the fact that $Q \leq N^d$, we obtain

$$(4.11) \quad \left| \widehat{\Phi}(k) \right| \leq \frac{N^\varepsilon}{Q},$$

if $k \neq 0$.

We now define that

$$K_{1,Q}(x, t) = \frac{1}{\widehat{\Phi}(0)} K_N(x, t) \Phi(t), \text{ and } K_{2,Q} = K_N - K_{1,Q}.$$

(4.6) follows immediately from Lemma 4.1 since intervals $J_{a/q} = [\frac{a}{q} + \frac{1}{100q^2}, \frac{a}{q} + \frac{1}{50q^2}]$'s are pairwise disjoint for all $Q \leq q \leq 5Q$ and $a \in \mathcal{P}_q$.

We now prove (4.7). In fact, represent Φ as its Fourier series to get

$$K_{2,Q}(x, t) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k \neq 0} \widehat{\Phi}(k) e^{2\pi i k t} K_N(x, t).$$

Thus its Fourier coefficient is

$$\widehat{K_{2,Q}}(n_1, n_2) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k \neq 0} \widehat{\Phi}(k) \mathbf{1}_{\{n_2 = n_1^d + k\}}(k).$$

Here $(n_1, n_2) \in \mathbb{Z}^2$ and $\mathbf{1}_A$ is the indicator function of a measurable set A . This implies that $\widehat{K_{2,Q}}(n_1, n_2) = 0$ if $n_2 = n_1^d$, and if $n_2 \neq n_1^d$,

$$\widehat{K_{2,Q}}(n_1, n_2) = -\frac{1}{\widehat{\Phi}(0)} \widehat{\Phi}(n_2 - n_1^d).$$

Applying (4.11), we estimate $\widehat{K_{2,Q}}(n_1, n_2)$ by

$$\left| \widehat{K_{2,Q}}(n_1, n_2) \right| \leq \frac{CN^\varepsilon}{Q},$$

since $N^{d-1} \leq Q \leq N^d$. Henceforth we obtain (4.7). Therefore we complete the proof. \square

Now we can state our theorem on the level set estimates.

Theorem 4.1. *For any positive numbers ε and $Q \geq N^{d-1}$, the level set defined as in (4.2) satisfies*

$$(4.12) \quad \lambda^2 |E_\lambda|^2 \leq C_1 N^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} |E_\lambda|^2 + \frac{C_2 N^\varepsilon}{Q} |E_\lambda|$$

holds for all $\lambda > 0$. Here C_1 and C_2 are constants independent of N and Q .

Proof. Notice that if $Q \geq N^d$, (4.12) becomes trivial since $E_\lambda = \emptyset$ if $\lambda \geq CN^{1/2}$. So we can assume that $N^{d-1} \leq Q \leq N^d$. For the function F_N and the level set E_λ given in (4.1) and (4.2) respectively, we define f to be

$$f(x, t) = \frac{F_N(x, t)}{|F_N(x, t)|} \mathbf{1}_{E_\lambda}(x, t).$$

Clearly

$$\lambda |E_\lambda| \leq \int_{\mathbb{T}^2} \overline{F_N(x, t)} f(x, t) dx dt.$$

By the definition of F_N , we get

$$\lambda |E_\lambda| \leq \sum_{n=-N}^N \overline{a_n} \widehat{f}(n, n^d).$$

Utilizing Cauchy-Schwarz's inequality, we have

$$\lambda^2 |E_\lambda|^2 \leq \sum_{n=-N}^N |\widehat{f}(n, n^d)|^2.$$

The right hand side can be written as

$$(4.13) \quad \langle K_N * f, f \rangle.$$

For any Q with $N^{d-1} \leq Q \leq N^d$, we employ Proposition 4.1 to decompose the kernel K_N . We then have

$$(4.14) \quad \lambda^2 |E_\lambda|^2 \leq |\langle K_{1,Q} * f, f \rangle| + |\langle K_{2,Q} * f, f \rangle|$$

From (4.6) and (4.7), we then obtain

$$(4.15) \quad \begin{aligned} \lambda^2 |E_\lambda|^2 &\leq C_1 N^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} \|f\|_1^2 + \frac{C_2 N^\varepsilon}{Q} \|f\|_2^2 \\ &\leq C_1 N^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} |E_\lambda|^2 + \frac{C_2 N^\varepsilon}{Q} |E_\lambda|, \end{aligned}$$

as desired. Therefore, we finish the proof of Theorem 4.1. \square

Corollary 4.1. *If $\lambda \geq 2C_1 N^{\frac{1}{2}-\frac{1}{2d}+\varepsilon}$, then*

$$(4.16) \quad |E_\lambda| \leq \frac{CN^{2^{d-1}-d+\varepsilon}}{\lambda^{2^d+2}}.$$

Here C_1 is the constant C_1 in Theorem 4.1 and C is a constant independent of N and λ .

Proof. Since $\lambda \geq 2C_1 N^{\frac{1}{2}-\frac{1}{2d}+\varepsilon}$, we simply take Q satisfies $2C_1 N^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} = \lambda^2$. Then Corollary 4.1 follows from Theorem 4.1. \square

Remark 4.1. *Corollary 4.1 is also true even if n^d in (4.1) is replaced by $n^d + P(n)$, where P is a polynomial in $\mathbb{Z}[x]$ whose degree is no more than $d-1$.*

We now are ready to finish the proof of Theorem 1.5. We only prove the case when d is odd. The even case can be done similarly by using $A_{d,4,N} \leq C$. In fact, let $p \geq (d-2)2^d + 6$ and write $\|F\|_p^p$ as

$$(4.17) \quad p \int_0^{2C_1 N^{\frac{1}{2} - \frac{1}{2^d} + \varepsilon}} \lambda^{p-1} |E_\lambda| d\lambda + p \int_{2C_1 N^{\frac{1}{2} - \frac{1}{2^d} + \varepsilon}}^{2N^{1/2}} \lambda^{p-1} |E_\lambda| d\lambda.$$

Observe that $A_{d,6,N} \leq N^\varepsilon$ implies

$$(4.18) \quad |E_\lambda| \leq \frac{N^\varepsilon}{\lambda^6}.$$

Thus the first term in (4.17) is bounded by

$$(4.19) \quad C N^{(\frac{1}{2} - \frac{1}{2^d})(p-6) + \varepsilon} \leq C N^{\frac{p}{2} - (d+1) + \varepsilon},$$

since $p \geq (d-2)2^d + 6$. From (4.16), the second term is majorized by

$$(4.20) \quad C N^{\frac{p}{2} - (d+1) + \varepsilon}.$$

Putting both estimates together, we complete the proof of Theorem 1.5.

5. A LOWER BOUND OF $K_{d,p,N}$

In this section we show that $N^{\frac{1}{2} - \frac{d+1}{p}}$ is the best upper bound of $K_{d,p,N}$ if $p \geq 2(d+1)$. Hence (1.7) can not be improved substantially, and it is sharp up to a factor of N^ε .

For $b \in \mathbb{N}$, let $S(N; b)$ be defined by

$$(5.1) \quad S(N; b) = \int_{\mathbb{T}^2} \left| \sum_{n=-N}^N e^{2\pi i t n^d + 2\pi i x n} \right|^{2b} dx dt.$$

Proposition 5.1. *Let $S(N; b)$ be defined as in (5.1). Then*

$$(5.2) \quad S(N; b) \geq C \left(N^b + N^{2b-(d+1)} \right).$$

Here C is a constant independent of N .

Proof. The proof is based on a standard argument in additive number theory. Clearly $S(N; b)$ is equal to the number of solutions of

$$(5.3) \quad \begin{cases} n_1 + \cdots + n_b = m_1 + \cdots + m_b \\ n_1^d + \cdots + n_b^d = m_1^d + \cdots + m_b^d \end{cases}$$

with $n_j, m_j \in \{-N, \dots, N\}$ for all $j \in \{1, \dots, b\}$. For each (m_1, \dots, m_b) , we may obtain a solution of (5.3) by taking $(n_1, \dots, n_b) = (m_1, \dots, m_b)$. Thus

$$(5.4) \quad S(N; b) \geq N^b.$$

To derive a further lower bound for $S(N; b)$, we set Ω to be

$$(5.5) \quad \Omega = \left\{ (x, t) : |x| \leq \frac{1}{60N}, \quad |t| \leq \frac{1}{60N^d} \right\}.$$

If $(x, t) \in \Omega$ and $|n| \leq N$, then

$$(5.6) \quad \left| tn^d + xn \right| \leq \frac{1}{30}.$$

Henceforth if $(x, t) \in \Omega$,

$$(5.7) \quad \left| \sum_{n=-N}^N e^{2\pi i t n^d + 2\pi i x n} \right| \geq \left| \operatorname{Re} \sum_{n=-N}^N e^{2\pi i t n^d + 2\pi i x n} \right| \geq \sum_{n=-N}^N \cos \left((2\pi t n^d + 2\pi x n) \right) \geq CN.$$

Consequently, we have

$$(5.8) \quad S(N; b) \geq \int_{\Omega} \left| \sum_{n=-N}^N e^{2\pi i t n^d + 2\pi i x n} \right|^{2b} dx dt \geq CN^{2b} |\Omega| \geq CN^{2b-(d+1)}.$$

□

Proposition 5.2. *Let $p \geq 2$ be even. Then $K_{d,p,N}$ satisfies*

$$(5.9) \quad K_{d,p,N} \geq C \left(1 + N^{\frac{1}{2} - \frac{d+1}{p}} \right).$$

Here C is a constant independent of N .

Proof. Let $p = 2b$ since p is even. Setting $a_n = 1$ for all n in the definition of $K_{d,p,N}$, we get

$$(5.10) \quad S(N; b) \leq K_{d,p,N}^p (2N)^b.$$

Consequently, by Proposition 5.1, we conclude (5.9). □

6. ESTIMATES OF $S(N; b)$

We have the following estimates for $S(N; b)$. The $d = 3$ case was proved by Hua. The method of Hua is different from what we utilize in this paper.

Theorem 6.1. *Let $S(N; b)$ be defined as in (5.1) and $d \geq 3$ be odd. Then*

$$(6.1) \quad S(N; b) \leq CN^{2b-(d+1)+\varepsilon}$$

holds provided $b \geq \max\{2^{d-1} + 1, 2^{d-2}(d-5) + 3\}$.

By Proposition 5.1, we see that the estimate (6.1) is (almost) sharp. The desired upper bound for $S(N; d+1)$ is not yet obtained. We now prove Theorem 6.1.

Proof. Let G_{λ} be the level set given by

$$(6.2) \quad G_{\lambda} = \{(x, t) \in \mathbb{T}^2 : |K_N(x, t)| \geq \lambda\}.$$

Here K_N is the function defined as in (4.5).

let $f = \mathbf{1}_{G_{\lambda}} K_N / |K_N|$ and we then have

$$(6.3) \quad \lambda |G_{\lambda}| \leq \sum_{n=-N}^N \widehat{f}(n, n^d) = \langle f_N, K_N \rangle,$$

where f_N is a rectangular Fourier partial sum defined by

$$(6.4) \quad f_N(x, t) = \sum_{\substack{|n_1| \leq N \\ |n_2| \leq N^d}} \widehat{f}(n_1, n_2) e^{2\pi i n_1 x} e^{2\pi i n_2 t}.$$

Employing Proposition 4.1 for K_N , we estimate the level set G_λ by

$$(6.5) \quad \lambda|G_\lambda| \leq |\langle f_N, K_{1,Q} \rangle| + |\langle f_N, K_{2,Q} \rangle|$$

for any $Q \geq N^{d-1}$. From (4.6) and (4.7), $\lambda|G_\lambda|$ can be bounded further by

$$(6.6) \quad C \left(N^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} \|f_N\|_1 + \sum_{\substack{|n_1| \leq N \\ |n_2| \leq N^d}} \left| \widehat{K_{2,Q}}(n_1, n_2) \widehat{f}(n_1, n_2) \right| \right).$$

Thus from the fact that L^1 norm of Dirichlet kernel D_N is comparable to $\log N$, (4.7), and Cauchy-Schwarz inequality, we have

$$(6.7) \quad \lambda|G_\lambda| \leq CN^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} |G_\lambda| + \frac{CN^{\frac{d+1}{2}+\varepsilon}}{Q} |G_\lambda|^{1/2},$$

for all $Q \geq N^{d-1}$. For $\lambda \geq 2CN^{1-2^{1-d}+\varepsilon}$, take Q to be a number satisfying

$$2CN^{-d2^{1-d}+1+\varepsilon} Q^{2^{1-d}} = \lambda$$

and then we obtain

$$(6.8) \quad |G_\lambda| \leq \frac{CN^{2^d-d+1}}{\lambda^{2^d+2}}.$$

Notice that

$$(6.9) \quad \|K_N\|_6 \leq N^{\frac{1}{2}} K_{d,6,N} \leq N^{\frac{1}{2}+\varepsilon}.$$

Henceforth by (6.3) we majorize $|G_\lambda|$ by

$$(6.10) \quad |G_\lambda| \leq \frac{CN^{3+\varepsilon}}{\lambda^6}.$$

For $b \geq 2^{d-1} + 1$, we now estimate $S(N; p)$ by

$$(6.11) \quad S(N; b) \leq C \int_{2CN^{1-2^{1-d}+\varepsilon}}^{2N} \lambda^{2b-1} |G_\lambda| d\lambda + C \int_0^{2CN^{1-2^{1-d}+\varepsilon}} \lambda^{2b-1} |G_\lambda| d\lambda.$$

From (6.8), the first term in the right hand side of (6.11) can be bounded by $CN^{2b-d-1+\varepsilon}$. From (6.10), the second term is clearly bounded by $N^{2b-d-1+\varepsilon}$. Putting both estimates together,

$$(6.12) \quad S(N; b) \leq CN^{2b-(d+1)+\varepsilon},$$

as desired. Therefore, we complete the proof. \square

7. ESTIMATES FOR THE NONLINEAR TERM

For any measurable function u on $\mathbb{T} \times \mathbb{R}$, we define the space-time Fourier transform by

$$(7.1) \quad \widehat{u}(n, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{T}} u(x, t) e^{-inx} e^{-i\lambda t} dx dt$$

and set

$$\langle x \rangle := 1 + |x|.$$

We now introduce the $X_{s,b}$ space, initially used by Bourgain.

Definition 7.1. Let I be an time interval in \mathbb{R} and $s, b \in \mathbb{R}$. Let $X_{s,b}(I)$ be the space of functions u on $\mathbb{T} \times I$ that may be represented as

$$(7.2) \quad u(x, t) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \widehat{u}(n, \lambda) e^{inx} e^{i\lambda t} d\lambda \quad \text{for } (x, t) \in \mathbb{T} \times I$$

with the space-time Fourier transform \widehat{u} satisfying

$$(7.3) \quad \|u\|_{X_{s,b}(I)} = \left(\sum_n \int \langle n \rangle^{2s} \langle \lambda + n^5 \rangle^{2b} |\widehat{u}(n, \lambda)|^2 d\lambda \right)^{1/2} < \infty.$$

Here the norm should be understood as a restriction norm.

We should take the time interval to be $[0, \delta]$ for a small positive number δ , and abbreviate $\|u\|_{X_{s,b}(I)}$ as $\|u\|_{s,b}$ for any function u restricted to $\mathbb{T} \times [0, \delta]$. We also define

$$(7.4) \quad \|u\|_{Y_s} := \|u\|_{s, \frac{1}{2}} + \left(\sum_n \langle n \rangle^{2s} \left(\int |\widehat{u}(n, \lambda)| d\lambda \right)^2 \right)^{\frac{1}{2}}.$$

Let ψ be a bump function supported in $[-2, 2]$ with $\psi(t) = 1, |t| \leq 1$, and let ψ_δ be

$$\psi_\delta(t) = \psi(t/\delta).$$

For any w which is a nonlinear function of u , the nonlinear operator \mathcal{N} is given by

$$(7.5) \quad \mathcal{N}u = -\psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau.$$

Lemma 7.1. The nonlinear term \mathcal{N} satisfies

$$(7.6) \quad \|\mathcal{N}u\|_{Y_s} \leq C \left(\|w\|_{s, -\frac{1}{2}} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \right),$$

where C is a constant independent of δ .

Proof. Represent w as its space-time inverse Fourier transform so that we write

$$(7.7) \quad \mathcal{N}u(x, t) = -\psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^5} \left(\sum_n \int \widehat{w}(n, \lambda) e^{inx} e^{i\lambda\tau} d\lambda \right) d\tau,$$

which is equal to

$$\begin{aligned} & -\psi_\delta(t) \sum_n \int \widehat{w}(n, \lambda) \int_0^t e^{-(t-\tau)(in)^5} e^{inx} e^{i\lambda\tau} d\tau d\lambda \\ & = -\psi_\delta(t) \sum_n \int \widehat{w}(n, \lambda) e^{inx} e^{-in^5 t} \frac{e^{i(\lambda+n^5)t} - 1}{i(\lambda + n^5)} d\lambda. \end{aligned}$$

We decompose the nonlinear term $\mathcal{N}u$ into three parts, denoted by $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ respectively.

$$\begin{aligned}
\mathcal{N}u(x, t) &= -\psi_\delta(t) \sum_n \int_{|\lambda+n^5| \leq \frac{1}{100\delta}} \widehat{w}(n, \lambda) e^{inx} e^{-in^5 t} \sum_{k \geq 1} \frac{(it)^k}{k!} (\lambda + n^5)^{k-1} d\lambda \\
&\quad + i\psi_\delta(t) \sum_n \int_{|\lambda+n^5| > \frac{1}{100\delta}} \frac{\widehat{w}(n, \lambda)}{\lambda + n^5} e^{inx} e^{i\lambda t} d\lambda \\
&\quad - i\psi_\delta(t) \sum_n \left(\int_{|\lambda+n^5| > \frac{1}{100\delta}} \frac{\widehat{w}(n, \lambda)}{\lambda + n^5} d\lambda \right) e^{inx} e^{-in^5 t} \\
&:= \mathcal{N}_1 u + \mathcal{N}_2 u + \mathcal{N}_3 u.
\end{aligned}$$

First we estimate \mathcal{N}_2 . Using Fourier series expansion for ψ , we get

$$\psi_\delta(t) = \sum_{m \in \mathbb{Z}} C_m e^{imt/\delta}.$$

Here the coefficients C_m 's satisfy

$$C_m \leq C(1 + |m|)^{-100}.$$

Hence $\mathcal{N}_2 u$ can be represent as

$$(7.8) \quad \mathcal{N}_2 u = i \sum_m C_m \sum_n e^{inx} \int_{|\lambda+n^5| > \frac{1}{100\delta}} \frac{\widehat{w}(n, \lambda)}{\lambda + n^5} e^{i(\lambda+m/\delta)t} d\lambda$$

By a change of variables $(\lambda + m/\delta) \mapsto \lambda$,

$$(7.9) \quad \mathcal{N}_2 u = i \sum_m C_m \sum_n e^{inx} \int_{|\lambda - \frac{m}{\delta} + n^5| > \frac{1}{100\delta}} \frac{\widehat{w}(n, \lambda - m/\delta)}{\lambda - \frac{m}{\delta} + n^5} e^{i\lambda t} d\lambda$$

Thus we estimate

$$(7.10) \quad \|\mathcal{N}_2 u\|_{s, \frac{1}{2}}^2 \leq C \sum_m (1 + |m|)^{-50} \sum_n \langle n \rangle^{2s} \int_{|\lambda - \frac{m}{\delta} + n^5| > \frac{1}{100\delta}} \frac{\langle \lambda + n^5 \rangle |\widehat{w}(n, \lambda - m/\delta)|^2}{|\lambda - \frac{m}{\delta} + n^5|^2} d\lambda.$$

Changing variables again, we obtain

$$(7.11) \quad \|\mathcal{N}_2 u\|_{s, \frac{1}{2}}^2 \leq C \sum_m (1 + |m|)^{-50} \sum_n \langle n \rangle^{2s} \int_{|\lambda+n^5| > \frac{1}{100\delta}} \frac{\langle \lambda + \frac{m}{\delta} + n^5 \rangle |\widehat{w}(n, \lambda)|^2}{\langle \lambda + n^5 \rangle^2} d\lambda.$$

Notice that $|\lambda + n^5| > \frac{1}{100\delta}$ implies

$$(7.12) \quad \langle \lambda + \frac{m}{\delta} + n^5 \rangle \leq 200m \langle \lambda + n^5 \rangle.$$

We obtain immediately

$$(7.13) \quad \|\mathcal{N}_2 u\|_{s, \frac{1}{2}} \leq C \|w\|_{s, -\frac{1}{2}}.$$

On the other hand,

$$\sum_n \langle n \rangle^{2s} \left(\int |\widehat{\mathcal{N}_2 u}(n, \lambda)| d\lambda \right)^2 \leq C \sum_m \langle m \rangle^{-5} \sum_n \langle n \rangle^{2s} \left(\int_{|\lambda - \frac{m}{\delta} + n^5| > \frac{1}{100\delta}} \frac{|\widehat{w}(n, \lambda - m/\delta)| d\lambda}{|\lambda - \frac{m}{\delta} + n^5|} \right)^2,$$

which is clearly bounded by

$$(7.14) \quad \sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)| d\lambda}{\langle \lambda + n^5 \rangle} \right)^2.$$

Putting (7.13) and (7.14) together, we have

$$(7.15) \quad \|\mathcal{N}_2 u\|_{Y_s} \leq C \left(\|w\|_{s, -\frac{1}{2}} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \right).$$

We now estimate \mathcal{N}_1 . Let A_n be defined by

$$(7.16) \quad A_n = \int_{|\lambda + n^5| \leq \frac{1}{100\delta}} \widehat{w}(n, \lambda) (\lambda + n^5)^{k-1} d\lambda.$$

Then $\mathcal{N}_1 u$ can be written as

$$(7.17) \quad \mathcal{N}_1 u(x, t) = - \sum_{k \geq 1} \frac{i^k}{k!} t^k \psi_\delta(t) \sum_n A_n e^{inx} e^{-in^5 t}.$$

Hence the space-time Fourier transform of $\mathcal{N}_1 u$ satisfies

$$(7.18) \quad \left| \widehat{\mathcal{N}_1 u}(n, \lambda) \right| \leq \sum_{k \geq 1} \frac{1}{k!} |A_n| \left| \mathcal{F}_{\mathbb{R}}(\tilde{\psi}_\delta)(\lambda + n^5) \right|,$$

where $\tilde{\psi}_\delta(t) = t^k \psi_\delta(t)$. Using the definition of Fourier transform, we have

$$\left| \mathcal{F}_{\mathbb{R}}(\tilde{\psi}_\delta)(\lambda + n^5) \right| \leq C \delta^{k+1} k^3 \langle \delta(\lambda + n^5) \rangle^{-3}.$$

Thus

$$\begin{aligned} \|\mathcal{N}_1 u\|_{Y_s}^2 &\leq \sum_{k \geq 1} \frac{C}{k^5} \sum_n \langle n \rangle^{2s} |A_n|^2 \delta^{2k} \int \delta^2 \langle \lambda + n^5 \rangle \langle \delta(\lambda + n^5) \rangle^{-6} d\lambda \\ &\quad + \sum_{k \geq 1} \frac{C}{k^5} \sum_n \langle n \rangle^{2s} |A_n|^2 \delta^{2k} \left(\int \delta \langle \delta(\lambda + n^5) \rangle^{-3} d\lambda \right)^2 \\ &\leq \sum_{k \geq 1} \frac{C}{k^5} \sum_n \langle n \rangle^{2s} |A_n|^2 \delta^{2k}. \end{aligned}$$

Clearly A_n is bounded by

$$(7.19) \quad |A_n| \leq C \delta^{-k} \int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda.$$

Henceforth, we obtain

$$(7.20) \quad \|\mathcal{N}_1 u\|_{Y_s} \leq C \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}}.$$

Similarly, we may obtain

$$(7.21) \quad \|\mathcal{N}_3 u\|_{Y_s} \leq C \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}}.$$

Therefore we complete the proof. □

8. LOCAL WELL-POSEDNESS OF (1.2)

We now start to derive the local well-posedness of (1.2). For this purpose, we only need to consider the well-posedness of the Cauchy problem:

$$(8.1) \quad \begin{cases} u_t + \partial_x^5 u + (u^k - \int_{\mathbb{T}} u^k dx) u_x = 0 \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, t \in \mathbb{R} \end{cases}.$$

Here $k \geq 2$ and we only need to consider the monomial case without loss of generality. This is because the gauge transform

$$(8.2) \quad u(x, t) := v \left(x - \int_0^t \int_{\mathbb{T}} v^k(y, \tau) dy d\tau, t \right)$$

can be used here for reducing the well-posedness problem of (1.2) to the well-posedness of (8.1). This gauge transform was employed in [4].

Let w be the nonlinear function defined by

$$(8.3) \quad w = \left(u^k - \int u^k dx \right) u_x.$$

We need the following estimate on the nonlinear function w , in order to establish a contraction on the space $\{u : \|u\|_{Y_s} \leq M\}$ for some $M > 0$. We postpone the proof of Proposition 8.1 to Section 9.

Proposition 8.1. *For $s > 1/2$, there exists $\theta > 0$ such that, for the nonlinear function w given by (8.3),*

$$(8.4) \quad \|w\|_{s, -\frac{1}{2}} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \leq C \delta^\theta \|u\|_{Y_s}^{k+1}.$$

Here C is a constant independent of δ and u .

By applying Duhamel principle, the corresponding integral equation associated to (8.1) is

$$(8.5) \quad u(x, t) = e^{-t\partial_x^5} \phi(x) - \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau,$$

where w is defined as in (8.3).

Since we are only seeking for the local well-posedness, we may use a bump function to truncate time variable. Then it suffices to find a local solution of

$$u(x, t) = \psi_\delta(t) e^{-t\partial_x^5} \phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau.$$

Let T be an operator given by

$$(8.6) \quad Tu(x, t) := \psi_\delta(t) e^{-t\partial_x^5} \phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau.$$

The first term (the linear term) and the second term (the nonlinear term) in (8.6) are denoted by $\mathcal{L}u$ and $\mathcal{N}u$, respectively. Henceforth Tu can be represented as $\mathcal{L}u + \mathcal{N}u$.

Lemma 8.1. *The linear term \mathcal{L} satisfies*

$$(8.7) \quad \|\mathcal{L}u\|_{Y_s} \leq C\|\phi\|_{H^s}.$$

Here C is a constant independent of δ .

Proof. Notice that

$$\widehat{\mathcal{L}u}(n, \lambda) = \widehat{\phi}(n)\mathcal{F}_{\mathbb{R}}\psi_\delta(\lambda + n^5) = \widehat{\phi}(n)\delta\mathcal{F}_{\mathbb{R}}\psi(\delta(\lambda + n^5)),$$

Thus from the definition of Y_s norm,

$$\begin{aligned} \|\mathcal{L}u\|_{Y_s} &= \left(\sum_n \int \langle n \rangle^{2s} \langle \lambda + n^5 \rangle \left| \widehat{\phi}(n)\delta\mathcal{F}_{\mathbb{R}}\psi(\delta(\lambda + n^5)) \right|^2 d\lambda \right)^{\frac{1}{2}} \\ &\quad + \left(\sum_n \langle n \rangle^{2s} \left(\int \left| \widehat{\phi}(n)\delta\mathcal{F}_{\mathbb{R}}\psi(\delta(\lambda + n^5)) \right| d\lambda \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since ψ is a Schwartz function, its Fourier transform is also a Schwartz function. Using the fast decay property for the Schwartz function, we have

$$\|\mathcal{L}u\|_{Y_s} \leq C \left(\sum_n \langle n \rangle^{2s} \left| \widehat{\phi}(n) \right|^2 \right)^{\frac{1}{2}} = C\|\phi\|_{H^s}.$$

□

Proposition 8.2. *Let $s > 1/2$ and T be the operator defined as in (8.6). Then there exists a positive number θ such that*

$$(8.8) \quad \|Tu\|_{Y_s} \leq C \left(\|\phi\|_{H^s} + \delta^\theta \|u\|_{Y_s}^{k+1} \right).$$

Here C is a constant independent of δ .

Proof. Since $Tu = \mathcal{L}u + \mathcal{N}u$, Proposition 8.2 follows from Lemma 8.1, Lemma 7.1 and Proposition 8.1. □

Proposition 8.2 yields that for δ sufficiently small, T maps a ball in Y_s into itself. Moreover, we write

$$\begin{aligned} &\left(u^k - \int_{\mathbb{T}} u^k dx \right) u_x - \left(v^k - \int_{\mathbb{T}} v^k dx \right) v_x \\ &= \left(u^k - \int_{\mathbb{T}} u^k dx \right) (u - v)_x + \left((u^k - v^k) - \int_{\mathbb{T}} (u^k - v^k) dx \right) v_x \end{aligned}$$

which equals to

$$(8.9) \quad \left(u^k - \int_{\mathbb{T}} u^k dx \right) (u - v)_x + \sum_{j=0}^{k-1} \left((u - v) u^{k-1-j} v^j - \int_{\mathbb{T}} (u - v) u^{k-1-j} v^j dx \right) v_x.$$

For $k+1$ terms in (8.9), repeating similar argument as in the proof of Proposition 8.1, one obtains, for $s > 1/2$,

$$(8.10) \quad \|Tu - Tv\|_{Y_s} \leq C\delta^\theta \left(\|u\|_{Y_s}^k + \sum_{j=1}^{k-1} \|u\|_{Y_s}^{k-1-j} \|v\|_{Y_s}^{j+1} \right) \|u - v\|_{Y_s}.$$

Henceforth, for $\delta > 0$ small enough, T is a contraction and the local well-posedness follows from Picard's fixed-point theorem.

9. PROOF OF PROPOSITION 8.1

From the definition of w in (8.3), we may write $\widehat{w}(n, \lambda)$ as

$$(9.1) \quad \sum_{\substack{m+n_1+\dots+n_k=n \\ n_1+\dots+n_k \neq 0}} m \int \widehat{u}(m, \lambda - \lambda_1 - \dots - \lambda_k) \widehat{u}(n_1, \lambda_1) \cdots \widehat{u}(n_k, \lambda_k) d\lambda_1 \cdots d\lambda_k.$$

By duality, there exists a sequence $\{A_{n,\lambda}\}$ satisfying

$$(9.2) \quad \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |A_{n,\lambda}|^2 d\lambda \leq 1,$$

and $\|w\|_{s, -\frac{1}{2}}$ is bounded by

$$(9.3) \quad \sum_{\substack{m+n_1+\dots+n_k=n \\ n_1+\dots+n_k \neq 0}} \int \frac{\langle n \rangle^s |m|}{\langle \lambda + n^5 \rangle^{\frac{1}{2}}} |\widehat{u}(m, \lambda - \lambda_1 - \dots - \lambda_k)| \cdot |\widehat{u}(n_1, \lambda_1)| \cdots |\widehat{u}(n_k, \lambda_k)| |A_{n,\lambda}| d\lambda_1 \cdots d\lambda_k d\lambda.$$

Since the $X_{s,b}$ is a restriction norm, we may assume that u is supported in $\mathbb{T} \times [0, \delta]$. Moreover, we may assume that $|\widehat{u}|^\vee$ is supported in a δ -sized time interval (see [7]). Without loss of generality we can also assume $|n_1| \geq |n_2| \geq \dots \geq |n_k|$.

The trouble occurs mainly because of the factor $|m|$ resulted from $\partial_x u$. The idea is that either the factor $\langle \lambda + n^5 \rangle^{-\frac{1}{2}}$ can be used to cancel $|m|$, or $|m|$ can be distributed to some of \widehat{u} 's. More precisely, we consider three cases.

$$(9.4) \quad |m| < 1000k^2 |n_2|;$$

$$(9.5) \quad 1000k^2 |n_2| \leq |m| \leq 100k |n_1|;$$

$$(9.6) \quad |m| > 100k |n_1|.$$

9.1. Case (9.4). This is the simplest case. In fact, in this case, it is easy to see that

$$(9.7) \quad \langle n \rangle^s |m| \leq C \langle n_1 \rangle^s \langle n_2 \rangle^{\frac{1}{2}} \langle m \rangle^{\frac{1}{2}}.$$

Let

$$(9.8) \quad F(x, t) = \sum_n \int \frac{|A_{n,\lambda}|}{\langle \lambda + n^5 \rangle^{\frac{1}{2}}} e^{i\lambda t} e^{inx} d\lambda;$$

$$(9.9) \quad G(x, t) = \sum_n \int \langle n \rangle^{\frac{1}{2}} |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda$$

$$(9.10) \quad H(x, t) = \sum_n \int \langle n \rangle^s |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda$$

$$(9.11) \quad U(x, t) = \sum_n \int |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda$$

Then using (9.7), we can estimate (9.3) by
(9.12)

$$C \sum_{m+n_1+\dots+n_k=n} \int \widehat{F}(n, \lambda) \widehat{G}(m, \lambda - \lambda_1 - \dots - \lambda_k) \widehat{H}(n_1, \lambda_1) \widehat{G}(n_2, \lambda_2) \prod_{j=3}^k \widehat{U}(n_j, \lambda_j) d\lambda_1 \dots d\lambda_k d\lambda,$$

which clearly equals

$$C \int_{\mathbb{T} \times \mathbb{R}} F(x, t) G(x, t)^2 H(x, t) U(x, t)^{k-2} dx dt.$$

Apply Hölder inequality to majorize it by

$$C \|F\|_4 \|G\|_{6+}^2 \|H\|_4 \|U\|_{6(k-2)-}^{k-2}.$$

Since U is supported on $\mathbb{T} \times [-2\delta, 2\delta]$, one more use of Hölder inequality yields

$$(9.13) \quad (9.3) \leq C \delta^\theta \|F\|_4 \|G\|_{6+}^2 \|H\|_4 \|U\|_{6(k-2)-}^{k-2}.$$

We list some useful local embedding facts on $X_{s,b}$.

$$(9.14) \quad X_{0, \frac{3}{10}} \subseteq L_{x,t}^4, \quad X_{0+, \frac{1}{2}+} \subseteq L_{x,t}^6, \quad (t \text{ local})$$

$$(9.15) \quad X_{\alpha, \frac{1}{2}} \subseteq L_{x,t}^q, \quad 0 < \alpha < \frac{1}{2}, \quad 2 \leq q < \frac{6}{1-2\alpha} \quad (t \text{ local}),$$

$$(9.16) \quad X_{\frac{1}{2}-\alpha, \frac{1}{2}-\alpha} \subseteq L_t^q L_x^r, \quad 0 < \alpha < \frac{1}{2}, \quad 2 \leq q, r < \frac{1}{\alpha}.$$

The two embedding results in (9.14) are consequences of the discrete restriction estimates on L^4 (Theorem 3.1) and L^6 (Theorem 1.4) respectively. (9.15) and (9.16) follow by interpolation. (9.14) yields

$$\|F\|_4 \leq C \|F\|_{0, \frac{3}{10}} \leq C \left(\sum_n \int |A_{n,\lambda}|^2 d\lambda \right)^{1/2} \leq C,$$

and

$$\|H\|_4 \leq C \|H\|_{0, \frac{3}{10}} \leq C \|u\|_{s, \frac{1}{2}} \leq C \|u\|_{Y_s}.$$

(9.15) implies

$$\|G\|_{6+} \leq C \|G\|_{0+, \frac{1}{2}} \leq C \|u\|_{s, \frac{1}{2}} \leq C \|u\|_{Y_s}.$$

Using (9.16), we get

$$\|U\|_{6(k-2)-} \leq C \|U\|_{\frac{1}{2}-, \frac{1}{2}-} \leq C \|u\|_{s, \frac{1}{2}} \leq C \|u\|_{Y_s}.$$

Henceforth, we have, for the case (9.4),

$$(9.17) \quad (9.3) \leq C \delta^\theta \|u\|_{Y_s}^{k+1}.$$

9.2. Case (9.5). In this case, we should further consider two subcases.

$$(9.18) \quad |m + n_1| \leq 1000k^2 |n_2|$$

$$(9.19) \quad |m + n_1| > 1000k^2 |n_2|$$

In the subcase (9.18), we use the triangle inequality to get

$$(9.20) \quad |n| = |m + n_1 + n_2 + \dots + n_k| \leq C |n_2|$$

Hence, we have

$$(9.21) \quad \langle n \rangle^s |m| \leq C \langle n_2 \rangle^s \langle m \rangle^{\frac{1}{2}} \langle n_1 \rangle^{\frac{1}{2}}.$$

Thus this subcase can be treated exactly the same as the case (9.4). We omit the details.

For the subcase (9.19), observe that

$$(9.22) \quad n^5 - (m^5 + n_1^5 + \cdots + n_k^5) = (m + n_1)^5 - m^5 - n_1^5 + B,$$

where B is given by

$$(9.23) \quad 5(m + n_1)^4 b + 10(m + n_1)^3 b^2 + 10(m + n_1)^2 b^3 + 5(m + n_1) b^4 + b^5.$$

Here $b = n_2 + \cdots + n_k$. Clearly we can estimate B by

$$(9.24) \quad |B| \leq 100k(m + n_1)^4 |n_2|.$$

On the other hand, notice that

$$(9.25) \quad (m + n_1)^5 - m^5 - n_1^5 = 5(m + n_1)mn_1(m^2 + n_1^2 + mn_1).$$

This implies

$$(9.26) \quad \begin{aligned} & |(m + n_1)^5 - m^5 - n_1^5| \\ & \geq \frac{15}{4} |m + n_1| |m| |n_1| \max\{|m|, |n_1|\}^2 \\ & \geq 90k^2 (m + n_1)^4 |n_2|. \end{aligned}$$

From (9.24) and (9.26), we get

$$(9.27) \quad |n^5 - (m^5 + n_1^5 + \cdots + n_k^5)| \geq C|m||n_1|^2 \langle n_2 \rangle \geq C|m|^3.$$

Henceforth, at least one of following statements must hold:

$$(9.28) \quad |\lambda + n^5| \geq C|m|^3,$$

$$(9.29) \quad |(\lambda - \lambda_1 - \cdots - \lambda_k) + m^5| \geq C|m|^3,$$

$$(9.30) \quad \exists i \in \{1, \dots, k\} \text{ such that } |\lambda_i + n_i^5| \geq C|m|^3.$$

For (9.28), (9.3) can be bounded by

$$(9.31) \quad \sum_{m+n_1+\cdots+n_k=n} \int \langle n_1 \rangle^s |\widehat{u}(m, \lambda - \lambda_1 - \cdots - \lambda_k)| |\widehat{u}(n_1, \lambda_1)| \cdots |\widehat{u}(n_k, \lambda_k)| |A_{n,\lambda}| d\lambda_1 \cdots d\lambda_k d\lambda.$$

Let F_1 be defined by

$$(9.32) \quad F_1(x, t) = \sum_n \int |A_{n,\lambda}| e^{i\lambda t} e^{inx} d\lambda.$$

Then we represent (9.31) as

$$(9.33) \quad \sum_{m+n_1+\cdots+n_k=n} \int \widehat{F}_1(n, \lambda) \widehat{U}(m, \lambda - \lambda_1 - \cdots - \lambda_k) \widehat{H}(n_1, \lambda_1) \prod_{j=2}^k \widehat{U}(n_j, \lambda_j) d\lambda_1 \cdots d\lambda_k d\lambda.$$

Here H and U are functions defined in (9.10) and (9.11) respectively. Clearly (9.33) equals

$$(9.34) \quad \int_{\mathbb{T} \times \mathbb{R}} F_1(x, t) H(x, t) U(x, t)^k dx dt.$$

Utilizing Hölder inequality, we estimate it further by

$$(9.35) \quad \|F_1\|_2 \|H\|_4 \|U\|_{4k}^k \leq C\delta^\theta \|u\|_{Y_s}^{k+1}.$$

This yields the desired estimate for the subcase (9.28).

For the subcase of (9.29), (9.3) is estimated by

$$\sum_{m+n_1+\dots+n_k=n} \int \frac{\langle n_1 \rangle^s |A_{n,\lambda}|}{\langle \lambda + n^5 \rangle^{\frac{1}{2}}} \langle (\lambda - \lambda_1 - \dots - \lambda_k) + m^5 \rangle^{\frac{1}{2}} |\widehat{u}(m, \lambda - \lambda_1 - \dots - \lambda_k)| \cdot |\widehat{u}(n_1, \lambda_1)| \dots |\widehat{u}(n_k, \lambda_k)| d\lambda_1 \dots d\lambda_k d\lambda,$$

which is equal to

$$(9.36) \quad \int_{\mathbb{T} \times \mathbb{R}} F(x, t) G(x, t) H(x, t) U^{k-1}(x, t) dx dt.$$

Apply Hölder inequality to control (9.36) by

$$(9.37) \quad \|F\|_4 \|G\|_4 \|H\|_4 \|U\|_{4(k-1)}^{k-1} \leq C \delta^\theta \|u\|_{Y_s}^{k+1}.$$

This completes the estimate for the subcase (9.29).

For the contribution of (9.30), we only consider $|\lambda_2 + n_2^5| \geq C|m|^3$ without loss of generality for $i \in \{2, \dots, k\}$. This is because the $|\lambda_1 + n^5| \geq C|m|^3$ case can be handled similarly as (9.29). Hence, in this case, (9.3) can be bounded by

$$\sum_{m+n_1+\dots+n_k=n} \int \frac{\langle n_1 \rangle^s |A_{n,\lambda}|}{\langle \lambda + n^5 \rangle^{\frac{1}{2}}} \langle \lambda_2 + n_2^5 \rangle^{\frac{1}{2}} |\widehat{u}(m, \lambda - \lambda_1 - \dots - \lambda_k)| \prod_{j=1}^k |\widehat{u}(n_j, \lambda_j)| d\lambda_1 \dots d\lambda_k d\lambda.$$

Now set a function I by

$$(9.38) \quad I(x, t) = \sum_n \langle \lambda + n^5 \rangle^{\frac{1}{2}} |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda.$$

Then we estimate (9.3) by

$$(9.39) \quad \int_{\mathbb{T} \times \mathbb{R}} F(x, t) H(x, t) I(x, t) U^{k-1}(x, t) dx dt,$$

which is majorized by

$$(9.40) \quad \|F\|_4 \|H\|_4 \|I\|_2 \|U\|_\infty^{k-1}.$$

Notice this time we cannot simply use Hölder's inequality to get δ as we did before because there is no way of making any above 4 or 2 even a little bit smaller. But this can be fixed as follows.

First observe that

$$\|u\|_{0,0} \leq \delta^{1/2} \|u\|_{L_x^2 L_t^\infty} \leq C \delta^{1/2} \|u\|_{0, \frac{1}{2}+},$$

for u is supported in a δ -sized interval in time variable. Thus by interpolation, we get

$$(9.41) \quad \|u\|_{0, \frac{3}{10}} \leq C \delta^{\frac{1}{5}-} \|u\|_{0, \frac{1}{2}}.$$

Since U can be assumed to be a function supported in a δ -sized time interval, we may put the same assumption to H . Henceforth, we have

$$(9.42) \quad \|H\|_4 \leq C \|H\|_{0, \frac{3}{10}} \leq C \delta^{\frac{1}{5}-} \|H\|_{0, \frac{1}{2}} \leq C \delta^{\frac{1}{5}-} \|u\|_{Y_s}.$$

Also note that

$$(9.43) \quad \|I\|_2 \leq \|u\|_{0, \frac{1}{2}} \leq \|u\|_{Y_s}.$$

and

$$(9.44) \quad \|U\|_\infty \leq C\|u\|_{Y_s}.$$

From (9.42), (9.43) and (9.44), we can estimate (9.3) by $C\delta^{\frac{1}{5}-}\|u\|_{Y_s}^{k+1}$ as desired. Therefore we finish our discussion for the case (9.5).

9.3. Case (9.6). In this case, let us further consider two subcases.

$$(9.45) \quad |m|^4 \leq 1000k^2|n_2|^4|n_3|$$

$$(9.46) \quad |m|^4 > 1000k^2|n_2|^4|n_3|$$

For the contribution of (9.45), we observe that from (9.45),

$$|m| \leq C|n_1|^{1/2}|n_2|^{1/2}|n_3|^{1/4},$$

since $|n_2| \leq |n_1|$. This implies immediately

$$(9.47) \quad \langle n \rangle^s |m| \leq C|m|^{s+1} \leq \langle m \rangle^s \langle n_1 \rangle^{1/2} \langle n_2 \rangle^{1/2} \langle n_3 \rangle^{1/4}.$$

Introduce a new function H_1 defined by

$$(9.48) \quad H_1(x, t) = \sum_n \int_{\mathbb{R}} \langle n \rangle^{1/4} |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda.$$

As before, in this case, we bound (9.3) by

$$(9.49) \quad \int_{\mathbb{T} \times \mathbb{R}} F(x, t) H(x, t) G^2(x, t) H_1(x, t) U^{k-3}(x, t) dx dt.$$

Then Hölder inequality yields

$$(9.50) \quad (9.3) \leq C\delta^\theta \|F\|_4 \|H\|_4 \|G\|_{6+}^2 \|H_1\|_{6+} \|U\|_\infty^{k-3}.$$

Clearly

$$\|H_1\|_{6+} \leq C\|u\|_{\frac{1}{4}+, \frac{1}{2}} \leq C\|u\|_{Y_s}.$$

Hence we obtain the desired estimate for the subcase (9.45).

We now turn to the contribution of (9.46). Clearly we have

$$(9.51) \quad \begin{aligned} & n^5 - (m^5 + n_1^5 + \cdots + n_k^5) \\ &= 5m^4(n_1 + b) + 10m^3(n_1 + b)^2 + 10m^2(n_1 + b)^3 + 5m(n_1 + b)^4 \\ & \quad + 5(n_1 + b)n_1b(n_1^2 + b^2 + n_1b) + O(n_2^4 n_3), \end{aligned}$$

since $|n_2| \geq |n_3| \geq \cdots \geq |n_k|$. From (9.6), (9.46), (9.51) and $n_1 + b \neq 0$, we have

$$(9.52) \quad |n^5 - (m^5 + n_1^5 + \cdots + n_k^5)| \geq C|m|^4.$$

This is similar to (9.27). Hence again we reduce the problems to (9.28), (9.29), and (9.30), which are all done in Subsection 9.2. Therefore we finish the case of (9.6).

Putting all cases together, we obtain

$$(9.53) \quad \|w\|_{s, -\frac{1}{2}} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}.$$

The desired estimate

$$(9.54) \quad \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}$$

can be obtained similarly, and we omit the details. Therefore we complete the proof of Proposition 8.1 by combining (9.53) and (9.54).

10. LOCAL WELL-POSEDNESS OF (1.3)

We now start to derive the local well-posedness of (1.3). Without loss of generality, we only need to consider the well-posedness of the Cauchy problem:

$$(10.1) \quad \begin{cases} u_t + \partial_x^5 u + u^k u_x u_x = 0 \\ u(x, 0) = \phi(x), \quad x \in \mathbb{T}, t \in \mathbb{R} \end{cases}.$$

Now let w be the nonlinear function defined by

$$(10.2) \quad w = u^k u_x u_x.$$

As before, we need the following estimate on the nonlinear function w , in order to establish a contraction on the space $\{u : \|u\|_{Y_s} \leq M\}$ for some $M > 0$. A proof of Proposition 10.1 will appear in Section 11.

Proposition 10.1. *For $s > 1$, there exists $\theta > 0$ such that, for the nonlinear function w given by (10.2),*

$$(10.3) \quad \|w\|_{s, -\frac{1}{2}} + \left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \leq C \delta^\theta \|u\|_{Y_s}^{k+2}.$$

Here C is a constant independent of δ and u .

By applying Duhamel principle, we reduce the problem to the well-posedness of corresponding integral equation associated to (10.1)

$$(10.4) \quad u(x, t) = e^{-t\partial_x^5} \phi(x) - \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau,$$

where w is defined as in (10.2). Using the local smooth truncation, we only need to seek a local solution of

$$u(x, t) = \psi_\delta(t) e^{-t\partial_x^5} \phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau.$$

Let T_1 be an operator given by

$$(10.5) \quad T_1 u(x, t) := \psi_\delta(t) e^{-t\partial_x^5} \phi(x) - \psi_\delta(t) \int_0^t e^{-(t-\tau)\partial_x^5} w(x, \tau) d\tau.$$

Proposition 10.2. *Let $s \geq 1$ and T_1 be the operator defined as in (10.5). Then there exists a positive number θ such that*

$$(10.6) \quad \|T_1 u\|_{Y_s} \leq C \left(\|\phi\|_{H^s} + \delta^\theta \|u\|_{Y_s}^{k+2} \right).$$

Here C is a constant independent of δ .

Proof. Since $Tu = \mathcal{L}u + \mathcal{N}u$, Proposition 10.2 follows from Lemma 8.1, Lemma 7.1 and Proposition 10.1. \square

Proposition 10.2 yields that for δ sufficiently small, T maps a ball in Y_s into itself. By a similar argument as in the proof of Proposition 10.1, one obtains, for $s \geq 1$,

$$(10.7) \quad \|T_1 u - T_1 v\|_{Y_s} \leq \delta^\theta C(\|u\|_{Y_s}, \|v\|_{Y_s}) \|u - v\|_{Y_s}.$$

Here $C(\|u\|_{Y_s}, \|v\|_{Y_s})$ is a real number depending on $\|u\|_{Y_s}$ and $\|v\|_{Y_s}$. Henceforth, for $\delta > 0$ sufficiently small, T_1 is a contraction and the local well-posedness follows from Picard's fixed-point theorem. This completes the proof of Theorem 1.1.

11. PROOF OF PROPOSITION 10.1

We only present details for proving

$$(11.1) \quad \|w\|_{s, -\frac{1}{2}} \leq C\delta^\theta \|u\|_{Y_s}^{k+2}.$$

The estimates for the extra term

$$\left(\sum_n \langle n \rangle^{2s} \left(\int \frac{|\widehat{w}(n, \lambda)|}{\langle \lambda + n^5 \rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \leq C\delta^\theta \|u\|_{Y_s}^{k+2}$$

are similar, and we omit the details. For simplicity, we assume $\delta = 1$.

From the definition of w in (10.2), we may write $\widehat{w}(n, \lambda)$ as

$$(11.2) \quad \sum_{m_1+m_2+n_1+\dots+n_k=n} m_1 m_2 \int \widehat{u}(m_1, \lambda - \mu - \lambda_1 - \dots - \lambda_k) \widehat{u}(m_2, \mu) \prod_{j=1}^k \widehat{u}(n_j, \lambda_j) d\mu d\lambda_1 \dots d\lambda_k.$$

Notice that

$$(11.3) \quad w = \frac{1}{k+1} \partial_x u^{k+1} \partial_x u.$$

Hence for free we can put additional conditions for $m_1, m_2, n_1, \dots, n_k$:

$$(11.4) \quad m_1 + n_1 + \dots + n_k \neq 0$$

and

$$(11.5) \quad m_2 + n_1 + \dots + n_k \neq 0.$$

Without loss of generality, we assume that

$$(11.6) \quad |m_1| \geq |m_2| \text{ and } |n_1| \geq \dots \geq |n_k|.$$

Henceforth, by duality, $\|w\|_{s, -\frac{1}{2}}$ is bounded by

$$(11.7) \quad \sum_{\substack{m_1+m_2+n_1+\dots+n_k=n \\ m_2+n_1+\dots+n_k \neq 0 \\ |m_1| \geq |m_2| \\ |n_1| \geq \dots \geq |n_k|}} \int \frac{\langle n \rangle^s |A_{n, \lambda}| |m_1| |m_2|}{\langle \lambda + n^5 \rangle^{\frac{1}{2}}} |\widehat{u}(m_1, \lambda - \mu - \lambda_1 - \dots - \lambda_k)| |\widehat{u}(m_2, \mu)| \\ \cdot |\widehat{u}(n_1, \lambda_1)| \dots |\widehat{u}(n_k, \lambda_k)| d\mu d\lambda_1 \dots d\lambda_k d\lambda.$$

Here the sequence $\{A_{n, \lambda}\}$ satisfying

$$(11.8) \quad \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |A_{n, \lambda}|^2 d\lambda \leq 1,$$

Carrying on the similar idea as before, we want to either distribute m_1, m_2 into some \widehat{u} 's or get some decay factor to cancel m_1 . More precisely, let us consider two cases.

$$(11.9) \quad |m_1 + m_2| \leq 1000k^2|n_1|;$$

$$(11.10) \quad |m_1 + m_2| > 1000k^2|n_1|;$$

11.1. Case (11.9). In this subcase, we have

$$(11.11) \quad |n|^s \leq C|n_1|^s$$

since $n = m_1 + m_2 + n_1 + \cdots + n_k$ and $|n_1| \geq \cdots \geq |n_k|$. Hence we may distribute n^s into $\widehat{u}(n_1, \lambda_1)$ so that (11.7) is estimated by

$$(11.12) \quad \int |F(x, t)| |G_1(x, t)|^2 |H(x, t)| |U(x, t)|^{k-1} dx dt,$$

where F, H , and U are functions defined as in (9.8), (9.10) and (9.11), respectively, and G_1 is given by

$$(11.13) \quad G_1(x, t) = \sum_n \int |n| |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda.$$

By a use of Hölder inequality and $s \geq 1$, we dominate (11.12) by

$$(11.14) \quad \|F\|_4 \|G_1\|_4^2 \|H\|_4 \|U\|_\infty^{k-1},$$

which is clearly bounded by $C\|u\|_{Y_s^{k+2}}^{k+2}$, as desired.

11.2. Case (11.10). In this case, we have

$$(11.15) \quad |m_1| \geq 500k^2|n_1|.$$

First we consider the subcase $|m_2| \leq |n_1|$. In this subcase, we get

$$(11.16) \quad \begin{aligned} & n^5 - (m_1^5 + m_2^5 + n_1^5 + \cdots + n_k^5) \\ &= 5m_1^4(n_1 + b_2) + 10m_1^3(n_1 + b_2)^2 + 10m_1^2(n_1 + b_2)^3 + 5m_1(n_1 + b_2)^4 \\ & \quad + 5(n_1 + b_2)n_1b_2(n_1^2 + b_2^2 + n_1b_2) + O(m_2^4n_2) + O(n_2^4m_2) + O(n_2^4n_3), \end{aligned}$$

where $b_2 = m_2 + n_2 + \cdots + n_k$. Since $m_2 + n_1 + \cdots + n_k \neq 0$, $n_1 + b_2 \neq 0$. Notice that in this case $|n_1| \ll |m_1|$. Then we have either

$$(11.17) \quad \max\{m_2^4|n_2|, n_2^4|m_2|, n_2^4|n_3|\} \geq \frac{1}{100}m_1^4$$

$$(11.18) \quad \text{or } |n^5 - (m_1^5 + m_2^5 + n_1^5 + \cdots + n_k^5)| \geq m_1^4.$$

(11.17) implies

$$(11.19) \quad \langle n \rangle^s |m_1| |m_2| \leq C |m_1|^s |m_1| |m_2| \leq C \max\{|n_1|^s |n_2|^{s/4} |m_1| |m_2|, |n_1|^{s/4} |n_2|^s |m_1| |m_2|\}.$$

Henceforth we estimate (11.7) by

$$(11.20) \quad \int |F(x, t)| |G_1(x, t)|^2 |H(x, t)| |H_2(x, t)| |U(x, t)|^{k-2} dx dt,$$

where H_2 is defined by

$$(11.21) \quad H_2(x, t) = \sum_n \int \langle n \rangle^{s/4} |\widehat{u}(n, \lambda)| e^{i\lambda t} e^{inx} d\lambda.$$

Using Hölder inequality, we have

$$(11.22) \quad \|F\|_4 \|G_1\|_6^2 \|H\|_4 \|H_2\|_6 \|U\|_\infty^{k-2} \leq C \|u\|_{Y_s^{k+2}},$$

since $s > 1$. This finishes the case of (11.17). If (11.18) holds, then one of the following statements must be true:

$$(11.23) \quad |\lambda + n^5| \geq m_1^4$$

$$(11.24) \quad |\lambda - \mu - \lambda_1 - \cdots - \lambda_k + m_1^5| \geq m_1^4$$

$$(11.25) \quad |\mu + m_2^5| \geq m_1^4$$

$$(11.26) \quad \exists j \in \{1, \dots, k\}, \quad |\lambda_j + n_j^5| \geq m_1^4$$

The cases (11.23), (11.24), (11.25) and (11.26) can be done similarly as the cases (9.28), (9.29) and (9.30). We omit the details. This completes the discussion on the subcase $|m_2| \leq |n_1|$.

We now turn to the subcase $|m_2| > |n_1|$. In this subcase, observe that

$$(11.27) \quad \begin{aligned} & n^5 - (m_1^5 + m_2^5 + n_1^5 + \cdots + n_k^5) \\ &= 5m_1^4(m_2 + b_1) + 10m_1^3(m_2 + b_1)^2 + 10m_1^2(m_2 + b_1)^3 + 5m_1(m_2 + b_1)^4 \\ & \quad + 5(m_2 + b_1)m_2b_1(m_2^2 + b_1^2 + m_2b_1) + O(n_1^4n_2) \\ &= 5(m_2 + b_1)m_1(m_1 + m_2 + b_1)(m_1^2 + (m_2 + b_1)^2 + m_1(m_2 + b_1)) \\ & \quad + 5(m_2 + b_1)m_2b_1(m_2^2 + b_1^2 + m_2b_1) + O(n_1^4n_2), \end{aligned}$$

where $b_1 = n_1 + \cdots + n_k$. Notice that, from (11.10),

$$(11.28) \quad \begin{aligned} & |5(m_2 + b_1)m_1(m_1 + m_2 + b_1)(m_1^2 + (m_2 + b_1)^2 + m_1(m_2 + b_1))| \\ & \geq 2000k^2|m_2 + b_1||m_1|^3\langle n_1 \rangle. \end{aligned}$$

Clearly we also have

$$(11.29) \quad |5(m_2 + b_1)m_2b_1(m_2^2 + b_1^2 + m_2b_1)| \leq 15k|m_2 + b_1||m_1|^3|n_1|.$$

Since $m_2 + b_1 \neq 0$, we reduce the problem to either

$$(11.30) \quad n_1^4|n_2| \geq \frac{1}{100}|m_1|^3\langle n_1 \rangle$$

$$(11.31) \quad \text{or} \quad |n^5 - (m_1^5 + m_2^5 + n_1^5 + \cdots + n_k^5)| \geq m_1^3.$$

Notice that from (11.30), we obtain

$$(11.32) \quad \langle n \rangle^s |m_1| |m_2| \leq C |m_1|^{s+1} |m_2| \leq |m_1| |m_2| |n_1|^s |n_2|^{s/3}.$$

Then the desired estimate follows by using the same method as in (11.19). The case (11.31) can be handled exactly the same as the case (11.18). Hence we complete the proof for the subcase $|m_2| > |n_1|$. Therefore the discussion on Case (11.10) is done.

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